

C4W14R

1. (a) Find the binomial expansion of

$$\frac{1}{(4+3x)^3}, \quad |x| < \frac{4}{3}$$

in ascending powers of x , up to and including the term in x^3 .
Give each coefficient as a simplified fraction.

(6)

In the binomial expansion of

$$\frac{1}{(4-9x)^3}, \quad |x| < \frac{4}{9}$$

the coefficient of x^2 is A .

- (b) Using your answer to part (a), or otherwise, find the value of A .
Give your answer as a simplified fraction.

(2)

$$a) (4+3x)^{-3} = 4^{-3} \left(1 + \frac{3}{4}x\right)^{-3} = \frac{1}{64} \left(1 + \frac{3}{4}x\right)^{-3}$$

$$= \frac{1}{64} \left[1 + (-3) \left(\frac{3}{4}x\right) + \frac{(-3)(-4)}{2} \left(\frac{3}{4}x\right)^2 + \frac{(-3)(-4)(-5)}{6} \left(\frac{3}{4}x\right)^3 \right]$$

$$= \frac{1}{64} - \frac{9}{256}x + \frac{27}{512}x^2 - \frac{135}{2048}x^3$$

b) replace $3x$ with $-9x$ in x^2 part of expansion

$$\frac{1}{64} \left[\frac{(-3)(-4)}{2} \left(\frac{-9x}{4}\right)^2 \right] = \frac{243}{512}x^2 \quad \therefore A = \frac{243}{512}$$

2. (i) Find

$$\int x \cos\left(\frac{x}{2}\right) dx \quad (3)$$

(ii) (a) Express $\frac{1}{x^2(1-3x)}$ in partial fractions.

(4)

(b) Hence find, for $0 < x < \frac{1}{3}$

$$\int \frac{1}{x^2(1-3x)} dx \quad (3)$$

$$i) \begin{cases} u = x & v = 2\sin\left(\frac{1}{2}x\right) \\ u' = 1 & v' = \cos\left(\frac{1}{2}x\right) \end{cases} \int uv' = uv - \int u'v$$

$$= 2x\sin\left(\frac{1}{2}x\right) - \int 2\sin\left(\frac{1}{2}x\right) dx$$

$$= 2x\sin\left(\frac{1}{2}x\right) + 4\cos\left(\frac{1}{2}x\right) + C$$

$$ii) \frac{1}{x^2(1-3x)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{1-3x} \quad \therefore 1 = A(x)(1-3x) + B(1-3x) + C(x^2)$$

$$x=0 \Rightarrow \underline{B=1} \quad x=\frac{1}{3} \Rightarrow \frac{1}{9}C=1 \quad \therefore \underline{C=9}$$

$$x=1 \Rightarrow 1 = -2A - 2B + C \quad 2A = 9 - 2 - 1 = 6 \quad \therefore \underline{A=3}$$

$$\Rightarrow \frac{3}{x} + \frac{1}{x^2} + \frac{9}{1-3x}$$

$$b) 3 \int \frac{1}{x} dx + \int x^{-2} dx - 3 \int \frac{-3}{-3x} dx$$

$$= 3 \ln x - \frac{1}{x} - 3 \ln(1-3x) + C$$

$$= 3 \ln\left(\frac{x}{1-3x}\right) - \frac{1}{x} + C$$

3. The number of bacteria, N , present in a liquid culture at time t hours after the start of a scientific study is modelled by the equation

$$N = 5000(1.04)^t, \quad t \geq 0$$

where N is a continuous function of t .

- (a) Find the number of bacteria present at the start of the scientific study. (1)

- (b) Find the percentage increase in the number of bacteria present from $t = 0$ to $t = 2$. (2)

Given that $N = 15000$ when $t = T$,

- (c) find the value of $\frac{dN}{dt}$ when $t = T$, giving your answer to 3 significant figures. (4)

a) $t=0 \Rightarrow N=5000$

b) $t=2 \Rightarrow N=5408 \Rightarrow \% \text{ Increase} = \frac{408}{5000} \times 100 = 8.16\%$

c) $15000 = 5000(1.04)^T \Rightarrow 3 = 1.04^T$

$$\ln(3) = T \ln(1.04) \Rightarrow T = 28.01102276 \dots$$

$$\frac{dN}{dt} = 5000(1.04)^t \times \ln(1.04)$$

when $t=T$ $\frac{dN}{dt} = 588.3106 \dots \approx \underline{588}$ (3sf)

$$b) \text{ Area} = \frac{1}{2}(\ln 2) [2.1333 + 2(1.4792 + 1.0079) + 0.6667]$$

$$= \underline{2.69}$$

$$c) u = 1 + 3e^{-x} \Rightarrow \frac{du}{dx} = -3e^{-x} \Rightarrow -\frac{1}{3} du = e^{-x} dx$$

$$\Rightarrow \frac{4}{3} \int \frac{1}{u^{\frac{1}{2}}} \times \frac{1}{3} du = \frac{4}{9} \int u^{-\frac{1}{2}} du = \frac{-4}{9} [2u^{\frac{1}{2}}]$$

$$\Rightarrow -\frac{8}{9} \sqrt{1+3e^{-x}} + C$$

$$\int_{-3\ln 2}^0 \frac{4e^{-x}}{3\sqrt{1+3e^{-x}}} dx = -\frac{8}{9} [\sqrt{1+3e^{-x}}]_{-3\ln 2}^0$$

$$= -\frac{8}{9} [(\sqrt{4}) - (\sqrt{1+3e^{3\ln 2}})]$$

$$= -\frac{8}{9} [2 - \sqrt{1+3 \times 2^3}] = -\frac{8}{9} [2 - 5] = \frac{8}{3}$$

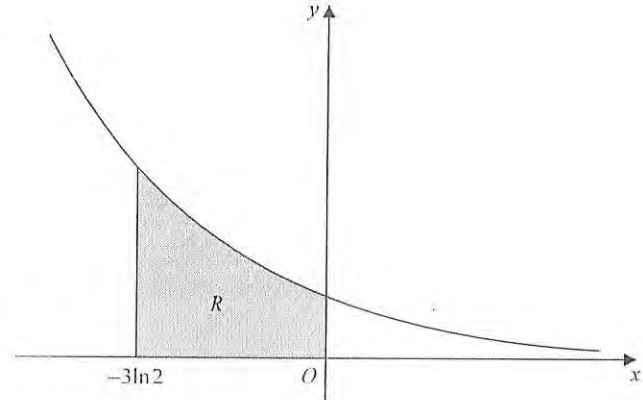


Figure 1

Figure 1 shows a sketch of part of the curve with equation $y = \frac{4e^{-x}}{3\sqrt{1+3e^{-x}}}$

The finite region R , shown shaded in Figure 1, is bounded by the curve, the x -axis, the line $x = -3\ln 2$ and the y -axis.

The table below shows corresponding values of x and y for $y = \frac{4e^{-x}}{3\sqrt{1+3e^{-x}}}$

x	$-3\ln 2$	$-2\ln 2$	$-\ln 2$	0
y	2.1333	1.4792	1.0079	0.6667

- (a) Complete the table above by giving the missing value of y to 4 decimal places. (1)
- (b) Use the trapezium rule, with all the values of y in the completed table, to obtain an estimate for the area of R , giving your answer to 2 decimal places. (3)
- (c) (i) Using the substitution $u = 1 + 3e^{-x}$, or otherwise, find

$$\int \frac{4e^{-x}}{3\sqrt{1+3e^{-x}}} dx$$

(5)

- (ii) Hence find the value of the area of R . (2)

5. Given that $y = 2$ at $x = \frac{\pi}{8}$, solve the differential equation

$$\frac{dy}{dx} = \frac{3y^2}{2\sin^2 2x}$$

giving your answer in the form $y = f(x)$.

(6)

$$\int \frac{1}{y^2} dy = \int \frac{3}{2} \operatorname{cosec}^2 2x$$

$$-\frac{1}{y} = -\frac{3}{4} \cot 2x + C$$

$$\therefore \frac{1}{y} = \frac{3}{4} \times \frac{1}{\tan 2x} + C$$

$$y = 2, x = \frac{\pi}{8} \quad \frac{1}{2} = \frac{3}{4} + C \quad \therefore C = -\frac{1}{4}$$

$$\frac{1}{y} = \frac{3}{4 \tan 2x} - \frac{1}{4} \times \frac{1}{\tan 2x} = \frac{3 - \tan 2x}{4 \tan 2x}$$

$$\therefore y = \frac{4 \tan 2x}{3 - \tan 2x}$$

6. Oil is leaking from a storage container onto a flat section of concrete at a rate of $0.48 \text{ cm}^3 \text{ s}^{-1}$. The leaking oil spreads to form a pool with an increasing circular cross-section. The pool has a constant uniform thickness of 3 mm.

Find the rate at which the radius r of the pool of oil is increasing at the instant when $r = 5 \text{ cm}$. Give your answer, in cm s^{-1} , to 3 significant figures.

(5)

\therefore Volume of cylinder increases at a rate
of $0.48 \text{ cm}^3/\text{s}$

$$\frac{dV}{dt} = 0.48 \quad V = \pi r^2 \times 0.3 \quad \Rightarrow \frac{dV}{dr} = 0.6\pi r$$

$$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt} = \frac{1}{0.6\pi r} \times 0.48$$

$$r = 5 \Rightarrow \frac{dr}{dt} = \frac{0.48}{3\pi} = \underline{\underline{0.0509}} \text{ (3sf)}$$

$$\therefore \cos t = \frac{5}{6} \quad \cos t = -\frac{1}{2} \quad t = \frac{2\pi}{3} \text{ (as previous)}$$

$$\therefore t = 0.5857^\circ$$

$$x = 2 \cos t = \frac{5}{3}$$

$$y = \sqrt{3} \cos 2t$$

$$y = \sqrt{3} (2 \cos^2 t - 1)$$

$$y = \sqrt{3} \left(2 \times \left(\frac{5}{6}\right)^2 - 1 \right)$$

$$y = \frac{7\sqrt{3}}{18} \quad \therefore Q \left(\frac{5}{3}, \frac{7\sqrt{3}}{18} \right)$$

7. The curve C has parametric equations

$$x = 2 \cos t, \quad y = \sqrt{3} \cos 2t, \quad 0 \leq t \leq \pi$$

where t is a parameter.(a) Find an expression for $\frac{dy}{dx}$ in terms of t .

(2)

The point P lies on C where $t = \frac{2\pi}{3}$ The line l is a normal to C at P .(b) Show that an equation for l is

$$2x - 2\sqrt{3}y - 1 = 0$$

(5)

The line l intersects the curve C again at the point Q .(c) Find the exact coordinates of Q .

You must show clearly how you obtained your answers.

(6)

$$\frac{dx}{dt} = -2 \sin t \quad \frac{dy}{dt} = -2\sqrt{3} \sin 2t$$

$$\therefore \frac{dy}{dx} = \frac{-2\sqrt{3} \sin 2t}{-2 \sin t} = \frac{\sqrt{3} (2 \sin t \cos t)}{\sin t} = 2\sqrt{3} \cos t$$

$$t = \frac{2\pi}{3} \quad x = 2 \cos \left(\frac{2\pi}{3} \right) = -1 \quad y = \sqrt{3} \cos \left(\frac{4\pi}{3} \right) = -\frac{\sqrt{3}}{2}$$

$$\frac{dy}{dx} = 2\sqrt{3} \cos \left(\frac{2\pi}{3} \right) = -\sqrt{3} = m_t \quad \therefore m_n = \frac{1}{\sqrt{3}}$$

$$\Rightarrow y + \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{3}}(x + 1) \Rightarrow \sqrt{3}y + \frac{3}{2} = x + 1 \quad (\times 2)$$

$$\Rightarrow 2x - 2\sqrt{3}y - 1 = 0 \quad \#$$

$$c) \quad 4 \cos t - 6 \cos 2t - 1 = 0 \quad 4(\cos t - 6(2 \cos^2 t - 1)) - 1 = 0$$

$$-12 \cos^2 t + 4 \cos t + 5 = 0 \Rightarrow 12 \cos^2 t - 4 \cos t - 5 = 0$$

$$(6 \cos t - 5)(2 \cos t + 1) = 0$$

8. With respect to a fixed origin O , the lines l_1 and l_2 are given by the equations

$$l_1: \mathbf{r} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \quad l_2: \mathbf{r} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ -2 \\ 5 \end{pmatrix}$$

where λ and μ are scalar parameters.

(a) Find, to the nearest 0.1° , the acute angle between l_1 and l_2 . (3)

The point A has position vector $\begin{pmatrix} 0 \\ 1 \\ 6 \end{pmatrix}$.

(b) Show that A lies on l_1 . (1)

The lines l_1 and l_2 intersect at the point X .

(c) Write down the coordinates of X . (1)

(d) Find the exact value of the distance AX . (2)

The distinct points B_1 and B_2 both lie on the line l_2 .

Given that $AX = XB_1 = XB_2$

(e) find the area of the triangle AB_1B_2 giving your answer to 3 significant figures. (3)

Given that the x coordinate of B_1 is positive,

(f) find the exact coordinates of B_1 and the exact coordinates of B_2 . (5)

$$a) \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -2 \\ 5 \end{pmatrix} = \frac{-5 - 4 + 5}{\sqrt{6}\sqrt{54}} = \frac{-4}{\sqrt{6}\sqrt{54}} \therefore \cos^{-1}\left(\frac{4}{18}\right)$$

$$\theta = \cos^{-1}\left(\frac{2}{9}\right) = \underline{\underline{77.2^\circ}}$$

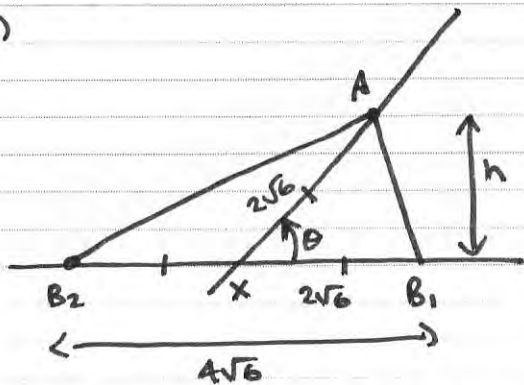
$$b) \mathbf{r} = \begin{pmatrix} 2-\lambda \\ -3+2\lambda \\ 4+\lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 6 \end{pmatrix} \text{ when } \lambda = 2 \therefore A \text{ lies on } l_1$$

$$c) X(2, -3, 4)$$

$$d) \vec{AX} = x - a = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ -2 \end{pmatrix}$$

$$\therefore |AX| = \sqrt{2^2 + 4^2 + 2^2} = \underline{2\sqrt{6}}$$

e)



$$h = 2\sqrt{6} \sin \theta$$

$$\cos \theta = \frac{2}{9} \quad \begin{array}{c} 9 \\ \triangle \theta \\ 2 \end{array} \quad \sqrt{77}$$

$$\therefore \sin \theta = \frac{\sqrt{77}}{9}$$

$$h = 2\sqrt{6} \times \frac{\sqrt{77}}{9}$$

$$\begin{aligned} \text{Area} &= \frac{1}{2}bh = 2\sqrt{6} \left(2\sqrt{6} \times \frac{\sqrt{77}}{9} \right) \\ &= \frac{8\sqrt{77}}{3} \approx \underline{23.4} \text{ (3sf)} \end{aligned}$$

$$f) x\vec{B}_1 = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ -2 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} = \mu \begin{pmatrix} 5 \\ -2 \\ 5 \end{pmatrix}$$

$$\left| \begin{pmatrix} 5 \\ -2 \\ 5 \end{pmatrix} \right| = 3\sqrt{6} \text{ but } x\vec{B}_1 = 2\sqrt{6} \therefore \mu = \frac{2}{3}$$

$$B_1 = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} + \frac{2}{3} \begin{pmatrix} 5 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 16/3 \\ -13/3 \\ 22/3 \end{pmatrix} \quad B_2 = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} + \frac{2}{3} \begin{pmatrix} 5 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} -4/3 \\ -5/3 \\ 2/3 \end{pmatrix}$$